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population and in political and social economy on which at present we are almost wholly in the dark. The mortality amongst emigrants and in the white and coloured populations, the effects of change of climate and occupation, of the quiet cultivation of land, of fevered life in cities, of the excitement of gold-mining or oil speculations, of early or late marriages, of people fleeing from poverty and misery to competence and sometimes sudden wealth, might all find some light thrown on them by a census so carefully conducted. It would be worthy of the wealth and enlightenment of the American nation to make the attempt, and it is not to be expected but that such a country would triumph over all the difficulties which would naturally attend it. These difficulties, we imagine, will stimulate rather than discourage them in effecting the improvements suggested, and in giving to their census the character which it ought to bear amongst the nations of the world.

On the Construction of Tables by the Method of Differences. By Peter Gray, F.R.A.S., *Honorary Member of the Institute of Actuaries.*

[Read before the Institute, 25th March, 1867.]

Section III.—*On the Construction of Tables in which the Characteristic Function is Rational and Integral.*

(85). We have now to apply the principles laid down in the previous sections to the construction of tables, first of rational, and secondly of irrational, functions. It is only formations of the class last mentioned that are of real importance. A little previous attention to the class first mentioned, however, is desirable, as affording an opportunity of elucidating, under elementary conditions, the arrangement of the work which is found most convenient in practice. The present section will, therefore, be devoted to the formation of successive values of rational functions.

(86). The operation by which successive values of a function are formed, when an initial value and its differences are given, is simply a reverting of that by which, when the requisite number of successive values is given, we thence deduce the differences. The differences being formed by subtraction, the successive values are built up, so to speak, by addition. Thus, take the example in (65), or rather its converse, and let the data be

$$u_x=1438,\ \Delta u_x=541,\ \Delta^2 u_x=120,\ \Delta^3 u_x=12,\ \text{constant};$$

then, arranging them in a line, we ascend to successive values of u_x as follows:—

Δ^3	Δ^2	Δ^1	u
12	120	541	1438
	132	661	1979
	144	793	2640
	156	937	3433
	168	1093	4370
	180	1261	5463
	⋮	⋮	⋮

The first column is not filled up, as it would consist of a mere repetition of the value at the top. The second is formed by continuous addition to the initial term of the constant third difference; and the third and fourth are formed in like manner by continuous addition to their initial terms of the successive terms in the preceding columns respectively.

(87). From the manner of formation just described it is easy to show that the nth term in

The first column is Δ^3
second ,, $\Delta^2+n\Delta^3$
third ,, $\Delta^1+n\Delta^2+\dfrac{n(n-1)}{2}\Delta^3$
fourth ,, $u_x+n\Delta^1+\dfrac{n(n-1)}{2}\Delta^2+\dfrac{n(n-1)(n-2)}{2.3}\Delta^3,$

which (70) is equal to u_{x+n}.

(88). The above may be generalized as follows:—If u_x be a function of m dimensions in x, then $\Delta^m u_x$ being constant, (60) we shall have—

$$\begin{aligned}
\Delta^m u_{x+n} &= \Delta^m u_x \\
\Delta^{m-1}u_{x+n} &= \Delta^{m-1}u_x+n\Delta^m u_x \\
\Delta^{m-2}u_{x+n} &= \Delta^{m-2}u_x+n\Delta^{m-1}u_x+\frac{n(n-1)}{2}\Delta^m u_x \\
\vdots\quad & \quad\vdots\qquad\vdots\qquad\vdots \\
\Delta^1 u_{x+n} &= \Delta^1 u_x+n\Delta^2 u_x+\frac{n(n-1)}{2}\Delta^3 u_x+\frac{n(n-1)(n-2)}{2.3}\Delta^4 u_x+\dots \\
\Delta^0 u_{x+n}, &
\end{aligned}$$

that is,

$$u_{x+n} = u_x+n\Delta^1 u_x+\frac{n(n-1)}{2}\Delta^2 u_x+\frac{n(n-1)(n-2)}{2.3}\Delta^3 u_x+\dots$$

or, more generally,

$$\Delta^r u_{x+n}=\Delta^r u_x+n\Delta^{r+1}u_x+\frac{n(n-1)}{2}\Delta^{r+2}u_x+\frac{n(n-1)(n-2)}{2.3}\Delta^{r+3}u_x\dots$$

the series being continued till the differences of u_x vanish.

We shall have occasion for these results hereafter.

(89). The operation in (86) may obviously be continued to any required extent. In its present form, however, it would soon become very irksome, and it is found expedient to arrange the work in a different manner. The arrangement adopted is influenced, of course, by the number of differences that have to be taken account of; and therefore, in exemplifying it, a commencement will be made with functions of the second degree, ascending, in this section, to those of the third.

(90). *Example* 1.—Given $u_x = 17x^2 + 22x - 15$; it is required to form a series of successive values of u_x when the initial value of x is 100.

The following are the preliminary operations :—

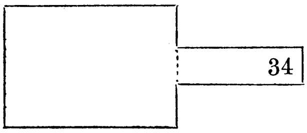
17	22	$-15(1$	17	22	$-15(100$
	39	24		1722	$172185 = u_{100}$
	56				
	34	39(1		34	39(100
		73			$3439 = \Delta u_{100}$
		$34 = \Delta^2 u_x = \Delta^2 u_{100}$			

On the left, using the operation in (62), we form the first and second differences of u_x, viz., $\Delta u_x = 34x + 39$, and $\Delta^2 u_x = 34$, constant; and on the right, 100, the initial value of x, is substituted in u_x and Δu_x, giving $u_{100} = 172185$, and $\Delta u_{100} = 3439$. (9). With these elements the required formation is effected, as follows :—

100	172185 3439	110	208105 3779	120	247425 4119
1	175624 3473	1	211884 3813	1	251544 4153
2	179097 3507	2	215697 3847	2	255697 4187
3	182604 3541	3	219544 3881	3	259884 4221
4	186145 3575	4	223425 3915	4	264105 4255
5	189720 3609	5	227340 3949	5	268360 4289
6	193329 3643	6	231289 3983	6	272649 4323
7	196972 3677	7	235272 4017	7	276972 4357
8	200649 3711	8	239289 4051	8	281329 4391
9	204360 3745	9	243340 4085	9	285720 4425
110	208105 3779	120	247425 4119	130	290145 4459

(91). Paper being prepared, and suitably ruled in columns, the successive values of the variable, 100, 101, &c., are filled in as shown. The first difference, 3439, is then inserted on the second line, and the successive terms of this series are formed on the alternate lines by continuous addition of the constant second difference, 34.

(92). This addition is best performed by aid of a card of the form here shown :—



It may be made of thin card or thick cartridge paper; and a slight degree of spring is communicated to the *tongue* by bending it back at the dotted line, to allow the card to be *slidden* down, after each addition, without smearing the newly-written figures. The tongue is pressed down with the fore finger of the left hand, to bring the figures to be added into apposition; and on its withdrawal after the addition is made, the tongue rises from the paper, and the card is slidden down for a new addition. Many applications of the blotting paper are thus saved.

(93). In using the card it is not necessary to form, by actual addition, more than the terminal portion of each term of the series of first differences: the formation in this manner may be restricted to the number of figures that there are on the card. In the present example this number is two; so that the operation with the card is simply the continuous addition of 34, neglecting carriages. We thus produce successively 73, 07, 41, 75, &c. These are then, by the information derived from the first term, 3439, rapidly and unerringly expanded, first into 473, 507, 541, &c., and finally into 3473, 3507, 3541, &c.

(94). It seems hardly necessary to point out how the additions are carried forward from one column to the next. An extra difference is formed, *in pencil*, at the bottom of each column, and is then transferred, in ink, to its place at the top of the next column. See the several columns in the example.

(95). The series of first differences having been completed to the extent required, the first term of the principal series, $u_{100}=172185$, is inserted in its place, and the remaining terms are formed by continuous addition.

(96). Now as to verification. In consequence of the mutual dependence of the terms, it will suffice to check the several series at intervals, by comparison of a term at each of the points selected for this purpose with the value of the same term formed by an independent process. The agreement of the terms so compared will afford sufficient presumption of the accuracy of the intermediate terms.

(97). If the series to be constructed be of no great extent, the verification values may be formed by Horner's process, (9). The following is the operation for the principal series when the verification values are ten terms apart:—

$$\begin{array}{lrl}
17 & 22 & -15(100 \\
 & 1722 & 172185 = u_{100} \\
 & 3422 & \\
\hline
17 & 3422 & 172185(10 \\
 & 3592 & 208105 = u_{110} \\
 & 3762 & \\
\hline
17 & 3762 & 208105(10 \\
 & 3932 & 247425 = u_{120} \\
 & \vdots & \vdots
\end{array}$$

and so on.

(98). But if the table to be formed is of considerable extent, it will be well to have recourse at once, for the construction of the verification values, to an operation analogous to that by which the series to be verified is formed. The operation now in view may be described as follows:—

(99). In differencing an algebraical function we arrive at a constant difference, the *order* of which depends on the dimensions of the function. This has been proved (60), and we have seen it to hold, in the case in which the constant difference of the variable is unit; but it is true also when the difference of the variable is any integer whatsoever. In this case we likewise arrive at a constant difference, and of the same order as in the first case. Hence it appears that we shall be able, by a process analogous to that in (90), to form a series the terms of which shall be equidistant terms of the series to be verified. The common interval between the terms will, of course, depend on the increment given to the variable.

(100). A slight extension of our notation will now be necessary. When the increment of x is unity, the successive differences of u_x are denoted by $\Delta^1 u_x$, $\Delta^2 u_x$, &c.; and when the increment is other than unity, this will be indicated by attaching the increment to the symbol Δ. Thus, if the increment of x be n, the successive differences will be denoted by $\Delta_n^1 u_x$, $\Delta_n^2 u_x$, &c.; and so also if the increment be 10, that is, if $n=10$, the differences will be denoted by $\Delta_{10}^1 u_x$, $\Delta_{10}^2 u_x$, &c.

(101). We will now verify the operation in (90) by forming the series whose terms shall be values of u_x corresponding to 100, 110, 120, &c. The requisite differences are formed as follows:—

$$\begin{array}{lll}
17 & 22 & -15(10 \\
 & 192 & 1905 \\
 & 362 & \\
\hline
 & 340 & 1920(10 \\
 & & 5320 \\
\cline{3-3}
 & & 3400=\Delta_{10}^2 u_{100}
\end{array}
\qquad
\begin{array}{ll}
340 & 1920(100 \\
 & 35920 \ =\Delta_{10}^1 u_{100}
\end{array}$$

The above operations correspond with the preliminary operations in (90). On the left, changing x into $x+10$, we form $\Delta_{10}^1 u_x=340x+1920$, and $\Delta^2 u_x=3400$;* and on the right, by the substitution of 100 for x in $\Delta_{10}^1 u_x$, we form $\Delta_{10}^1 u_{100}=35920$. (9). The differences thus formed being used as in (90), we have the operation in the margin, and the terms formed are terms situated at intervals of ten in the series to be verified.

100	172185 35920
10	208105 39320
20	247425 42720
30	290145 46120
40	336265 49520
50	385785 52920
60	438705 56320
70	495025 59720
80	554745 63120
90	617865 66520
200	684385

(102). This series may itself be verified as follows:—

$$\begin{array}{lll}
17 & 22 & -15(100 \\
 & 1722 & 172185=u_{100} \\
 & 3422 & \\
\hline
17 & 3422 & 172185(100 \\
 & 5122 & 684385=u_{200} \\
 & \vdots &
\end{array}$$

(103). The series of first differences in the operation of (90) may be verified, ere commencing the final additions, by aid of the formula in (88),

$$\Delta^1 u_{x+n}=\Delta^1 u_x+n\Delta^2 u_x;$$

but it is unnecessary to exhibit the process.

---

* The last step here may well be omitted, since it consists in adding 340 × 10 to 1920, and then subtracting 1920 from the sum. The difference of a function of one dimension is, of course, the product of the first coefficient by the increment of x.

(104). In all cases the verification values ought to be formed first of all, and inserted in their places, so as to avoid interruption to the continuity of the main operations.

(105). *Example* 2.—To form a series of successive values of $u_x = 486298 + 32060x - 12x^2$, the initial value of x being 10.

−12	32060	486298(1	−12	32060	486298(10
	32048	518346		31940	$805698 = u_{10}$
	32036				
	−24	32048		−24	32048(10
		$-24 = \Delta^2 u_x$			$31808 = \Delta^1 u_{10}$

10	805698 31808	20	1122698 31568
1	837506 31784	1	1154266 31544
2	869290 31760	2	1185810 31520
3	901050 31736	3	1217330 31496
4	932786 31712	4	1248826 31472
5	964498 31688	5	1280298 31448
6	996186 31664	6	1311746 31424
7	1027850 31640	7	1343170 31400
8	1059490 31616	8	1374570 31376
9	1091106 31592	9	1405946 31352
20	1122698	30	1437298

In this example $\Delta^2 u_x$ is negative, and in consequence the formation of the series of first differences seems to consist of a succession of subtractions of 24. The series is more conveniently formed, however (the last two figures of it, at least), by successive addition of 76, the complement of 24 to a unit of the next higher denomination, and the preceding figures are easily filled in by inspection.

(106). It is worth while to remark that $\Delta^1 u_x$ being in a state of diminution, it will ultimately become negative. At the same point u_x will begin to decrease, and will also ultimately become negative. The first of these changes marks the passage of x through a value which renders u_x a maximum, and the second marks the passage of x through a root of $u_x=0$.

(107). *Example* 3.—To form a series of successive values of $u_x=6x^3+13x^2-22x+17$, the initial value of x being 10.

$$\begin{array}{cccc} 6 & 13 & -22 & 17(1 \\ & 19 & -3 & 14 \\ & 25 & 22 & \\ & 31 & & \\ \cline{2-4} & 18 & 44 & -3(1 \\ & & 62 & 59 \\ & & 80 & \\ \cline{3-4} & & 36 & 62(1 \\ & & & 36=\Delta^3u_x \end{array} \qquad \begin{array}{cccc} 6 & 13 & -22 & 17(10 \\ & 73 & 708 & 7097=u_{10} \\ & & & \\ & 18 & 44 & -3(10 \\ & & 224 & 2237=\Delta^1u_{10} \\ & & & \\ & & 36 & 62(10 \\ & & & 422=\Delta^2u_{10} \end{array}$$

$$\begin{aligned} \Delta^1u_x &= 18x^2+44x-3 \\ \Delta^2u_x &= 36x+62 \\ \Delta^3u_x &= 36 \end{aligned}$$

(108). The above is the preliminary work for the formation of the differences, &c.; and the following shows the construction of the first thirty terms :—

10	2237	20	8077	30	17517
1	422 2659	1	782 8859	1	1142 18659
2	458 3117	2	818 9677	2	1178 19837
3	494 3611	3	854 10531	3	1214 21051
4	530 4141	4	890 11421	4	1250 22301
5	566 4707	5	926 12347	5	1286 23587
6	602 5309	6	962 13309	6	1322 24909
7	638 5947	7	998 14307	7	1358 26267
8	674 6621	8	1034 15341	8	1394 27661
9	710 7331	9	1070 16411	9	1430 29091
20	746 8077	30	1106 17517	40	1466 30557

10	7097 2237	20	52777 8077	30	173057 17517
1	9334 2659	1	60854 8859	1	190574 18659
2	11993 3117	2	69713 9677	2	209233 19837
3	15110 3611	3	79390 10531	3	229070 21051
4	18721 4141	4	89921 11421	4	250121 22301
5	22862 4707	5	101342 12347	5	272422 23587
6	27569 5309	6	113689 13309	6	296009 24909
7	32878 5947	7	126998 14307	7	320918 26267
8	38825 6621	8	141305 15341	8	347185 27661
9	45446 7331	9	156646 16411	9	374846 29091
20	52777	30	173057	40	403937

The work consists of two portions, of which the second is a repetition of the first (with certain changes), not necessary in practice. On the line corresponding to u_{11}, $\Delta^2 u_{10}=422$ is set down; and the succeeding terms of the series of second differences, 458, 494, &c., are formed, *in pencil,* by continuous addition (by aid of a card) of the constant third difference, 36, the last term being $\Delta^2 u_{39}=1466$, as may be easily verified by aid of the proper formula of (88). The first difference, $\Delta^1 u_{10}=2237$, is now inserted, and by continuous addition, *in ink,* the entire series of first differences is formed, down to $\Delta^1 u_{40}=30557$, which may be verified as before. The pencilled second differences, for which there is no further occasion, are now to be rubbed out,* and $u_{10}=7097$ being inserted in its place, u_{11}, u_{12}, &c., are formed in succession by continuous addition. The completed operation is shown on the right, it being understood that in practice this is all that is exhibited. It is formed *upon* the first skeleton, the successive values of u_x occupying the lines left vacant by the erasure of the second differences.

(109). The operation may be verified by the formation of selected terms, either in series, in the manner of (101), or by Horner's process, thus :—

$$\begin{array}{rrrl} 6 & 13 & -22 & 17(40 \\ & 253 & 10098 & 403937=u_{40} \end{array}$$

(110). *Example* 4.—Given four consecutive values of a function of the third degree, as below. It is required to continue the series.

It will be perceived that this example differs from those that precede in that here the data are the requisite number of consecutive values, and the form of the function is not given.

	Δ^1+	Δ^2-	Δ^3+
$u_0 = 3289586$	289674	9246	18
$u_1 = 3579260$	280428	9228	
$u_2 = 3859688$	271200		
$u_3 = 4130888$			

The elements for the construction then are—

$$\begin{aligned} u_0 &= 3289586 \\ \Delta^1 u_0 &= 289674 \\ \Delta^2 u_0 &= -9246 \\ \Delta^3 u_0 &= 18 \end{aligned}$$

in regard to which it is to be observed that $\Delta^2 u_0$ is negative. This gives rise, of course, to a modification in the working. The formation of the first twenty terms is as follows:—

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* I may mention here that I find the best way of removing extensive pencil writing is to rub it, both before and after using the India-rubber, with a piece of flannel or other woollen cloth. The first application of the woollen, supposing the pencil used to be a *good* one, leaves little for the India-rubber to do, and the second most effectually removes the *greasy* feel that always remains more or less after the use of India-rubber, and restores the paper to its pristine state.

0	289674	10	198024
1	9246 280428	1	9066 188958
2	9228 271200	2	9048 179910
3	9210 261990	3	9030 170880
4	9192 252798	4	9012 161868
5	9174 243624	5	8994 152874
6	9156 234468	6	8976 143898
7	9138 225330	7	8958 134940
8	9120 216210	8	8940 126000
9	9102 207108	9	8922 117078
10	9084 198024	20	8904 108174

0	3289586 289674	10	5772416 198024
1	3579260 280428	1	5970440 188958
2	3859688 271200	2	6159398 179910
3	4130888 261990	3	6339308 170880
4	4392878 252798	4	6510188 161868
5	4645676 243624	5	6672056 152874
6	4889300 234468	6	6824930 143898
7	5123768 225330	7	6968828 134940
8	5349098 216210	8	7103768 126000
9	5565308 207108	9	7229768 117078
10	5772416	20	7346846

(111). Here $\Delta^2 u_0 = -9246$ is first inserted, *in pencil*, in the place to be finally occupied by u_1, and the succeeding second differences may then be formed by the continuous *subtraction* of $\Delta^3 u_0 = 18$, $\Delta^2 u_0$ and $\Delta^3 u_0$ having opposite signs. But a preferable mode of formation is by the continuous addition, by means of a card (92), of 82, the complement of 18 to 100, attending only to the last two figures, 28, 10, 92, &c. The preceding figures can then be filled in by inspection. And verification is obtained by (88), thus:—

$$\begin{aligned}\Delta^2 u_{19} &= \Delta^2 u_0 + 19\Delta^3 u_0 \\ &= -9246 + 19 \times 18 = -8904.\end{aligned}$$

The first difference of u_0, 289674, is now inserted in its place, and, the signs being again opposite, the remaining terms of this series would seem to require to be formed by subtraction. This inconvenience (for such most, if not all, computers would feel it) may, however, be avoided, as follows:—Form, by the proper formula of (88), a remote term of the series, say $\Delta^1 u_{20}$, thus:—

$$\begin{aligned}\Delta^1 u_{20} &= \Delta^1 u_0 + 20\Delta^2 u_0 + 190\Delta^3 u_0 \\ &= 289674 - 20 \times 9246 + 3420 \times 18 \\ &= 108174.\end{aligned}$$

This term will fall to be inserted immediately under $\Delta^2 u_{19}=8904$; and, *commencing at the bottom,* the first differences may now be formed by addition. After a very little practice it will be found nearly as easy to add *upwards* as *downwards.* It is convenient in this operation to place a piece of blotting paper under the hand, moving it upwards after the formation of every two or three terms. The pencilled second differences are now to be erased, and $u_0=3289586$ being inserted in its place, the required formation is completed by continuous addition. Verification is obtained as follows (88):—

$$\begin{aligned} u_{20} &= u_0+20\Delta^1 u_0+190\Delta^2 u_0+1140\Delta^3 u_0 \\ &= 3289586+20\times 289674-190\times 9246+1140\times 18 \\ &= 7346846. \end{aligned}$$

As in Example 3, I repeat the scheme of the operation, to show the appearance it assumes when the work is finished.

(112). I have now gone as far as is necessary in regard to the construction of rational functions. I have entered on the subject only on account of the opportunity it affords for illustrating the arrangement of the materials operated upon which is found most conducive to facility and perspicuity; and I have gone as far as three orders of differences, which are as many as can be conveniently dealt with in a single column. The mode of dealing with four and five orders will receive ample illustration in the concluding section, which will be occupied with the subject of irrational functions.

(113). To avoid misconception, I desire here to say, that in the actual construction of tables of rational functions it will seldom, if ever, be necessary to have recourse to the formal methods I have exemplified. By careful investigation relations will generally be found to subsist, which will enable the desired construction to be effected by the expenditure of much less time and labour than would be requisite if the methods in question were made use of. Illustration of this remark is afforded by Mr. Laundy's *Table of Quarter Squares,* the methods employed in the construction of which are described by the author in his Introduction, pp. ix. to xiv.

(114). Further illustration is afforded by the specimen hereto appended, which I extract from the *Mechanics' Magazine,* vol. xxii., p. 104. It is one of two pages there given as a specimen of a Table of Squares, by Mr. David Booth, who dates his communication, "Charlotte Street, Bloomsbury, Nov. 5, 1834." It is well deserving of further publicity, on account of the ingenuity of its

Roots.	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.	Terms.
3600	1 296	18 496	55 696	112 896	190 096	287 296	404 496	541 696	698 896	876 096	0000
01	296	498	700	902	104	306	508	710	912	114	7201
02	297	501	705	909	113	317	521	725	929	133	4404
03	298	504	710	916	122	328	534	740	946	152	1609
04	298	506	714	922	130	338	546	754	962	170	8816
05	299	509	719	929	139	349	559	769	979	189	6025
06	300	512	724	936	148	360	572	784	996	208	3236
07	301	515	729	943	157	371	585	799	699 013	227	0449
08	301	517	733	949	165	381	597	813	029	245	7664
09	302	520	738	956	174	392	610	828	046	264	4881
3610	1 303	18 523	55 743	112 963	190 183	287 403	404 623	541 843	699 063	876 283	2100
11	303	525	747	969	191	413	635	857	079	301	9321
12	304	528	752	976	200	424	648	872	096	320	6544
13	305	531	757	983	209	435	661	887	113	339	3769
14	306	534	762	990	218	446	674	902	130	358	0996
15	306	536	766	996	226	456	686	916	146	376	8225
16	307	539	771	113 003	235	467	699	931	163	395	5456
17	308	542	776	010	244	478	712	946	180	414	2689
18	308	544	780	016	252	488	724	960	196	432	9924
19	309	547	785	023	261	499	737	975	213	451	7161
3620	1 310	18 550	55 790	113 030	190 270	287 510	404 750	541 990	699 230	876 470	4400
21	311	553	795	037	279	521	763	542 005	247	489	1641
22	311	555	799	043	287	531	775	019	263	507	8884
23	312	558	804	050	296	542	788	034	280	526	6129
24	313	561	809	057	305	553	801	049	297	545	3376
25	314	564	814	064	314	564	814	064	314	564	0625
26	314	566	818	070	322	574	826	078	330	582	7876
27	315	569	823	077	331	585	839	093	347	601	5129
28	316	572	828	084	340	596	852	108	364	620	2384
29	316	574	832	090	348	606	864	122	380	638	9641
3630	1 317	18 577	55 837	113 097	190 357	287 617	404 877	542 137	699 397	876 657	6900
31	318	580	842	104	366	628	890	152	414	676	4161
32	319	583	847	111	375	639	903	167	431	695	1424
33	319	585	851	117	383	649	915	181	447	713	8689
34	320	588	856	124	392	660	928	196	464	732	5956
35	321	591	861	131	401	671	941	211	481	751	3225
36	322	594	866	138	410	682	954	226	498	770	0496
37	322	596	870	144	418	692	966	240	514	788	7769
38	323	599	875	151	427	703	979	255	531	807	5044
39	324	602	880	158	436	714	992	270	548	826	2321
3640	1 324	18 604	55 884	113 164	190 444	287 724	405 004	542 284	699 564	876 844	9600
41	325	607	889	171	453	735	017	299	581	863	6881
42	326	610	894	178	462	746	030	314	598	882	4164
43	327	613	899	185	471	757	043	329	615	901	1449
44	327	615	903	191	479	767	055	343	631	919	8736
45	328	618	908	198	488	778	068	358	648	938	6025
46	329	621	913	205	497	789	081	373	665	957	3316
47	330	624	918	212	506	800	094	388	682	976	0609
48	330	626	922	218	514	810	106	402	698	994	7904
49	331	629	927	225	523	821	119	417	715	877 013	5201

arrangement, and the facility thereby afforded for the construction of the table, and its continuation to any extent that may be desired.

(115). The page given contains 500 squares. The last column, headed *Terms.* (contraction for *Terminations*), contains on each line the four figures which are the common termination of all the squares upon that line. With this understanding the column headed 0 contains the fifty squares $(3600)^2$ to $(3649)^2$, the column headed 1 contains the squares $(13600)^2$ to $(13649)^2$, the column headed 2 the squares $(23600)^2$ to $(23649)^2$, and so on.

(116). Now as to the mode of construction, which is conducted in the present form. The first difference of x^2 being $2x+1$, when x is 3600 this is 7201; and the constant second difference being 2, the terminal figures of column 0 are formed by continuous addition to 0000, the terminal figures of $(3600)^2$, of the terms of the series 7201, 7203, 7205, &c. (neglecting carriages). The series thus formed, occupying the last column, each term of it belongs equally to all the squares on the same line; for, the roots of each two adjoining squares differing from each other by 10,000, it is obvious that x^2 and $(x+10{,}000)^2$ have the last four figures common. Now mark the progression in the vertical columns. In column 0 the increase is by 0 and 1, according as there is an increase or a decrease in the leading figures of the last column. In column 1 the increase is by 2 and 3, subject to the same condition; in column 2 the increase is by 4 and 5; and so on up to column 9, in which the increase is by 18 and 19.

(117). It thus appears that a table of squares admits of being formed in this manner with great facility. It would, however, require verification. This would be obtained by addition of the several columns, the expression for the sum of a number of consecutive squares being well known.

On the Value of Annuities payable Half-yearly, Quarterly, &c. By Thomas Bond Sprague, M.A., *Actuary of the Equity and Law Life Assurance Society.* (*Part III.*)

In order to complete the consideration of this question as far as it relates to a single life, I will now give some numerical examples showing the magnitude of the corrections, or the additions to the value of annuities when payable half-yearly, as found from the formulæ of Mr. Baily and Mr. Woolhouse. Those formulæ will also be exhibited under some new aspects.